Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 21: Binomial Coefficients. Section 6.4

1 Binomial Coefficients

The binomial theorem gives the coefficients of the expansion of powers of binomial expressions.

Example 1. The expansion of $(x+y)^3$ can be found using combinatorial reasoning instead of multiplying the three terms out. When $(x+y)^3 = (x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added. Terms of the form x^3, x^2y, xy^2, y^3 arise. To obtain a term of the form x^3, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^3 term in the product has a coefficient of $1 = \binom{3}{3}$. To obtain a term of the form x^2y, an x must be chosen in two of the three sums. Hence, the number of such terms is the number of 2-combinations of three objects, or $\binom{3}{2}$. In the same way we get $\binom{3}{1}$ for the coefficient of xy^2 and $1 = \binom{3}{0}$ for the term y^3.

Theorem 2. *(THE BINOMIAL THEOREM) Let x and y be variables, and let n be a nonnegative integer. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Some properties of the binomial coefficients are

1. $\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$
2. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n \binom{n}{n} = 0.$

Theorem 3. *(PASCAL'S IDENTITY) Let n and k be positive integers with $n \geq k$. Then*

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$